Commuting semigroups of holomorphic mappings

Mark Elin
Department of Mathematics
ORT Braude College
P.O. Box 78, 21982 Karmiel, Israel
e-mail: mark.elin@gmail.com

Marina Levenshtein
Department of Mathematics
The Technion — Israel Institute of Technology
32000 Haifa, Israel
e-mail: marlev@list.ru

Simeon Reich
Department of Mathematics
The Technion — Israel Institute of Technology
32000 Haifa, Israel
e-mail: sreich@tx.technion.ac.il

David Shoikhet
Department of Mathematics
ORT Braude College
P.O. Box 78, 21982 Karmiel, Israel
e-mail: davs27@netvision.net.il

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Abstract

Let $S_1 = \{F_t\}_{t\geq 0}$ and $S_2 = \{G_t\}_{t\geq 0}$ be two continuous semigroups of holomorphic self-mappings of the unit disk $\Delta = \{z : |z| < 1\}$ generated by f and g, respectively. We present conditions on the behavior of f (or g) in a neighborhood of a fixed point of S_1 (or S_2), under which the commutativity of two elements, say, F_1 and G_1 of the semigroups implies that the semigroups commute, i.e., $F_t \circ G_s = G_s \circ F_t$ for all $s, t \geq 0$. As an auxiliary result, we show that the existence of the (angular or unrestricted) n-th derivative of the generator f of a semigroup $\{F_t\}_{t\geq 0}$ at a boundary null point of f implies that the corresponding derivatives of F_t, $t \geq 0$, also exist, and we obtain formulae connecting them for $n = 2, 3$.

1 Introduction

We denote by $\mathrm{Hol}(\Delta, D)$ the set of all holomorphic functions on the unit disk $\Delta = \{z : |z| < 1\}$ which map Δ into a domain $D \subset \mathbb{C}$, and by $\mathrm{Hol}(\Delta)$ the set of all holomorphic self-mappings of Δ.

We say that a family $S = \{F_t\}_{t\geq 0} \subset \mathrm{Hol}(\Delta)$ is **a one-parameter continuous semigroup on** Δ (a semigroup, in short) if

(i) $F_t(F_s(z)) = F_{t+s}(z)$ for all $t, s \geq 0$

and

(ii) $\lim\limits_{t\to 0^+} F_t(z) = z$ for all $z \in \Delta$.

If all the elements F_t, $t \geq 0$, of a semigroup S are automorphisms of Δ, then S can be extended to **a group** of automorphisms $\{F_t\}_{t\in\mathbb{R}}$ and property (i) holds for all real s and t.

It follows from a result of E. Berkson and H. Porta [4] that each semigroup is differentiable with respect to $t \in \mathbb{R}^+ = [0, \infty)$. So, for each one-parameter continuous semigroup $S = \{F_t\}_{t\geq 0} \subset \mathrm{Hol}(\Delta)$, the limit

$$\lim_{t\to 0^+} \frac{z - F_t(z)}{t} = f(z), \quad z \in \Delta,$$

exists and defines a holomorphic mapping $f \in \mathrm{Hol}(\Delta, \mathbb{C})$. This mapping f is called the **(infinitesimal) generator of** $S = \{F_t\}_{t\geq 0}$. Moreover, the function $u(t, z) := F_t(z)$, $(t, z) \in \mathbb{R}^+ \times \Delta$, is the unique solution of the Cauchy

problem

$$
\begin{cases} \dfrac{\partial u(t,z)}{\partial t} + f(u(t,z)) = 0, \\ u(0,z) = z, \quad z \in \Delta. \end{cases} \tag{1}
$$

This solution is univalent on Δ (see [1]).

We say that $\tau \in \overline{\Delta}$ is a fixed point of $F \in \mathrm{Hol}(\Delta)$ if either $F(\tau) = \tau$, where $\tau \in \Delta$, or $\lim\limits_{r\to 1^-} F(r\tau) = \tau$, where $\tau \in \partial\Delta = \{z : |z| = 1\}$. If F is not an automorphism of Δ with an interior fixed point, then by the Schwarz–Pick Lemma and the Julia–Wolff–Carathéodory Theorem, there is a unique fixed point $\tau \in \overline{\Delta}$ such that for each $z \in \Delta$, $\lim\limits_{n\to\infty} F_n(z) = \tau$, where the n-th iteration F_n of F is defined by $F_1 = F$, $F_n = F \circ F_{n-1}$, $n = 2, 3, \ldots$. Moreover, if $\tau \in \Delta$, then $|F'(\tau)| < 1$, and if $\tau \in \partial\Delta$, then the so-called angular derivative at the point τ (see the definition below) $F'(\tau) \in (0, 1]$. This point is called the Denjoy–Wolff point of F. The mapping F is of

— **dilation type**, if $\tau \in \Delta$,

— **hyperbolic type**, if $\tau \in \partial\Delta$ and $0 < F'(\tau) < 1$,

— **parabolic type**, if $\tau \in \partial\Delta$ and $F'(\tau) = 1$.

The mappings of parabolic type fall into two subclasses:

— **automorphic type**, if all orbits $F_n(z)$ are separated in the hyperbolic Poincaré metric ρ of Δ, i.e., $\lim\limits_{n\to\infty} \rho(F_n(z), F_{n+1}(z)) > 0$ for all $z \in \Delta$;

— **nonautomorphic type**, if no orbit $F_n(z)$ is hyperbolically separated, i.e., $\lim\limits_{n\to\infty} \rho(F_n(z), F_{n+1}(z)) = 0$ for all $z \in \Delta$.

Consider a semigroup $S = \{F_t\}_{t\geq 0}$ generated by $f \in \mathrm{Hol}(\Delta, \mathbb{C})$. It is a well-known fact that all elements F_t $(t > 0)$ of S are of the same type (dilation, hyperbolic or parabolic) and have the same Denjoy–Wolff point τ which is a null point (interior or boundary) of f. (Recall that $\tau \in \partial\Delta$ is a boundary null point of $f \in \mathrm{Hol}(\Delta, \mathbb{C})$ if $\lim\limits_{r\to 1^-} f(r\tau) = 0$.) If f generates a semigroup of dilation type (which does not consist of automorphisms), then $\mathrm{Re} f'(\tau) > 0$. In the hyperbolic case the angular derivative $f'(\tau)$ defined by $f'(\tau) := \lim\limits_{r\to 1^-} \frac{f(r\tau)}{(r-1)\tau}$ exists and is a positive real number; in the parabolic case $f'(\tau) = 0$ (see, for example, [20]).

We say that a function $f \in \mathrm{Hol}(\Delta, \mathbb{C})$ has an angular limit L at a point $\tau \in \partial\Delta$ and write $L := \angle\lim\limits_{z\to\tau} f(z)$, if $f(z) \to L$ as $z \to \tau$ in each Stolz angle $D_{\tau,\alpha} = \{z \in \Delta : |\arg(1 - \overline{\tau} z)| < \alpha\}$, $\alpha \in (0, \frac{\pi}{2})$. If L is finite and the

angular limit

$$M := \angle\lim_{z\to\tau} \frac{f(z) - L}{z - \tau}$$

exists, then M is said to be the angular derivative $f'(\tau)$.

It is known (see [16], p. 79) that the existence of the first angular derivative $f'(\tau)$ of a function $f \in \mathrm{Hol}(\Delta, \mathbb{C})$ is equivalent to each of the following conditions:

(1) there exists $\angle\lim\limits_{z\to\tau} f'(z)$, and then $f'(\tau) = \angle\lim\limits_{z\to\tau} f'(z)$;

(2) the function f admits the representation

$$f(z) = a_0 + a_1(z - \tau) + \gamma(z),$$

where $\gamma \in \mathrm{Hol}(\Delta, \mathbb{C})$, $\angle\lim\limits_{z\to\tau} \frac{\gamma(z)}{z-\tau} = 0$, and then $f'(\tau) = a_1$.

In Section 2 of this paper we show that higher order angular derivatives of f can also be defined by either one of these ways and the definitions are equivalent (Proposition 2). Furthermore, we show that for a semigroup $\{F_t\}_{t\geq 0}$ generated by $f \in \mathrm{Hol}(\Delta, \mathbb{C})$, the existence of the n-th ($n > 1$) angular derivative $f^{(n)}(\tau)$ of f at its boundary null point $\tau \in \partial\Delta$ implies that for each element F_t of the semigroup, the n-th angular derivative at τ also exists, and obtain formulae connecting $F^{(n)}(\tau)$ with $f^n(\tau)$ for $n = 2, 3$ (Theorem 1).

Using these facts, we investigate in Sections 3, 4, and 5 conditions under which the commutativity of two given elements of the semigroups $S_1 = \{F_t\}_{t\geq 0}$ and $S_2 = \{G_t\}_{t\geq 0}$ implies that the semigroups commute for the dilation, hyperbolic and parabolic cases, respectively (Theorems 2, 3, and 4).

2 Higher order boundary derivatives

We begin by recalling the following known fact.

Proposition 1 ([16], p. 79) *Let h be holomorphic in Δ. If $\mathrm{Im}\, h(z)$ has a finite angular limit at $\tau \in \partial\Delta$, then $(z - \tau)h'(z)$ has the angular limit 0 at τ.*

Proposition 2 *Let $f \in \mathrm{Hol}(\Delta, \mathbb{C})$ and let $\tau \in \partial\Delta$. Then the following assertions are equivalent for any integer $k \geq 0$:*

(i) The function f admits the representation

$$f(z) = \sum_{j=0}^{k} \frac{a_j}{j!}(z - \tau)^j + \gamma_k(z), \tag{2}$$

where $\angle \lim\limits_{z\to\tau} \frac{\gamma_k(z)}{(z-\tau)^k} = 0.$

(ii) The angular limit

$$\angle \lim_{z\to\tau} f^{(k)}(z)$$

exists finitely and coincides with a_k *in representation (2).*

(iii) For each $0 \le n \le k$*, the angular limit*

$$\angle \lim_{z\to\tau} f^{(n)}(z)$$

exists finitely and coincides with a_n *in representation (2).*

Proof.

(i)⇒(ii). Let (i) hold. We show by induction that for all $0 \le n \le k$, the following equality is satisfied:

$$f^{(n)}(z) - \sum_{j=0}^{k-n} \frac{a_{n+j}}{j!}(z-\tau)^j = \gamma_{k-n}(z), \text{ where } \angle \lim_{z\to\tau} \frac{\gamma_{k-n}(z)}{(z-\tau)^{k-n}} = 0. \quad (3)$$

It follows from this equality with $n = k$ that $f^{(k)}(z) - a_k = \gamma_0(z)$, and, hence, $\angle \lim\limits_{z\to\tau} f^k(z) = a_k$, as required.

For $n = 0$ relation (3) is obviously equivalent to (2). Suppose that it holds for $n = m-1$ ($m \le k$), i.e.,

$$f^{(m-1)}(z) - \sum_{j=0}^{k-m} \frac{a_{m-1+j}}{j!}(z-\tau)^j = \frac{a_k}{(k-m+1)!}(z-\tau)^{k-m+1} + \gamma_{k-m+1}(z). \quad (4)$$

Denote

$$h(z) := \frac{f^{(m-1)}(z) - \sum_{j=0}^{k-m} \frac{a_{m-1+j}}{j!}(z-\tau)^j}{(z-\tau)^{k-m+1}}.$$

Then there exists the finite angular limit

$$\angle \lim_{z\to\tau} h(z) = \frac{a_k}{(k-m+1)!}. \quad (5)$$

Now we find

$$(z-\tau)h'(z) = \frac{f^{(m)}(z) - \sum_{j=0}^{k-m-1} \frac{a_{m+j}}{j!}(z-\tau)^j}{(z-\tau)^{k-m}} - (k-m+1)h(z).$$

Since by Proposition 1, $\angle\lim\limits_{z\to\tau}(z-\tau)h'(z)=0$, we can write

$$\frac{f^{(m)}(z)-\sum_{j=0}^{k-m-1}\frac{a_{m+j}}{j!}(z-\tau)^j}{(z-\tau)^{k-m}}=(k-m+1)h(z)+\mu(z),$$

where $\angle\lim\limits_{z\to\tau}\mu(z)=0$. It follows from this equality and (5), that

$$\frac{f^{(m)}(z)-\sum_{j=0}^{k-m-1}\frac{a_{m+j}}{j!}(z-\tau)^j}{(z-\tau)^{k-m}}=\frac{a_k}{(k-m)!}+\gamma(z),$$

where $\angle\lim\limits_{z\to\tau}\gamma(z)=0$. Therefore,

$$f^{(m)}(z)-\sum_{j=0}^{k-m}\frac{a_{m+j}}{j!}(z-\tau)^j=\gamma_{k-m}(z),$$

where $\gamma_{k-m}(z):=\gamma(z)\cdot(z-\tau)^{k-m}$ and, consequently, $\angle\lim\limits_{z\to\tau}\frac{\gamma_{k-m}(z)}{(z-\tau)^{k-m}}=0$. In other words, (3) holds for $n=m$.
(ii)⇒(iii). Suppose now that there exists the finite limit

$$a_k:=\angle\lim_{z\to\tau}f^{(k)}(z). \tag{6}$$

Consider the equality

$$f^{(k-1)}(z)=f^{(k-1)}(0)+\int_0^z f^{(k)}(s)ds,\ \ z\in\Delta.$$

Since the angular limit (6) exists finitely, the function $f^k(z)$ is continuous on each curve $\Gamma(t),\ \alpha\le t\le\beta,\ \Gamma(\alpha)=0,\ \Gamma(\beta)=\tau$, strictly inside each Stolz angle at τ. Hence, there exists the finite angular limit

$$a_{k-1}:=\angle\lim_{z\to\tau}f^{(k-1)}(z)=f^{(k-1}(0)+\int_0^\tau f^{(k)}(s)ds.$$

Similarly, for each $0\le n\le k$, the limit

$$a_n:=\angle\lim_{z\to\tau}f^{^{(n)}}(z) \tag{7}$$

exists finitely.

(iii)$\Rightarrow$(i). Now we show by induction that for each $0 \le n \le k$,

$$f^{(k-n)}(z) = \sum_{j=0}^{n} \frac{a_{k-n+j}}{j!}(z-\tau)^j + \gamma_n(z) \tag{8}$$

with $\angle \lim\limits_{z\to\tau} \frac{\gamma_n(z)}{(z-\tau)^n} = 0$.

For $n = 0$ equality (8) follows immediately from (6). Suppose that it holds for $n = m-1$ ($m \le k$), i.e.,

$$f^{(k-m+1)}(z) = \sum_{j=0}^{m-1} \frac{a_{k-m+1+j}}{j!}(z-\tau)^j + \gamma_{m-1}(z), \tag{9}$$

where $\angle \lim\limits_{z\to\tau} \frac{\gamma_{m-1}(z)}{(z-\tau)^{m-1}} = 0$.

It is clear that

$$\frac{f^{(k-m)}(z) - a_{k-m}}{z-\tau} = \int_0^1 f^{(k-m+1)}(t\tau + (1-t)z)dt.$$

Therefore, by (7),

$$\angle \lim_{z\to\tau} \frac{f^{(k-m)}(z) - a_{k-m}}{z-\tau} = \angle \lim_{z\to\tau} \int_0^1 f^{(k-m+1)}(t\tau + (1-t)z)dt = a_{k-m+1}.$$

On the other hand, by (9),

$$\frac{f^{(k-m)}(z) - a_{k-m}}{z-\tau} = \int_0^1 f^{(k-m+1)}(t\tau + (1-t)z)dt =$$

$$= \int_0^1 \left(\sum_{j=0}^{m-1} \frac{a_{k-m+1+j}}{j!}(t\tau + (1-t)z - \tau)^j + \gamma_{m-1}(t\tau + (1-t)z) \right) dt =$$

$$= \sum_{j=0}^{m-1} \frac{a_{k-m+1+j}}{(j+1)!}(z-\tau)^j + \int_0^1 \gamma_{m-1}(t\tau + (1-t)z)dt.$$

Hence,

$$f^{(k-m)}(z) = \sum_{j=0}^{m} \frac{a_{k-m+j}}{j!}(z-\tau)^j + \gamma_m(z),$$

where $\gamma_m(z) = (z-\tau)\int_0^1 \gamma_{m-1}(t\tau + (1-t)z)dt$.

Now we verify that $\angle \lim\limits_{z\to\tau} \frac{\gamma_m(z)}{(z-\tau)^m} = 0$. Indeed,

$$\angle \lim_{z\to\tau} \frac{\gamma_m(z)}{(z-\tau)^m} = \angle \lim_{z\to\tau} \int_0^1 \frac{\gamma_{m-1}(t\tau + (1-t)z)}{(z-\tau)^{m-1}} dt =$$

$$= \angle \lim_{z\to\tau} \int_0^1 \frac{\gamma_{m-1}(t\tau + (1-t)z)}{(t\tau + (1-t)z - \tau)^{m-1}} \cdot \frac{(t\tau + (1-t)z - \tau)^{m-1}}{(z-\tau)^{m-1}} dt =$$

$$= \int_0^1 \left((1-t)^{m-1} \angle \lim_{z\to\tau} \frac{\gamma_{m-1}(t\tau + (1-t)z)}{(t\tau + (1-t)z - \tau)^{m-1}} \right) dt = 0,$$

and for $n = m$ (8) is proved. By induction, (8) holds for all $0 \le n \le k$. This equality with $n = k$ yields representation (2). ■

Remark 1 *It follows from the proof that Proposition 1 also holds if we replace the angular limit* $\angle \lim\limits_{z\to\tau}$ *by the unrestricted limit* $\lim\limits_{\substack{z\to\tau \\ z\in\Delta}}$ *in (i)–(iii).*

Remark 2 *Proposition 1 can be rephrased in terms of continuous extension of the higher order derivatives of* f *to* $\Delta \cup \{\tau\}$ *([5]).*

Let F be a holomorphic self-mapping of Δ and let $\tau \in \partial\Delta$ be a boundary fixed point of F. Then by the Julia–Wolff–Carathéodory Theorem, the first angular derivative $F'(\tau)$ either exists finitely and is a positive real number or equals infinity. If $\{F_t\}_{t\ge 0}$ is a one-parameter continuous semigroup with a boundary fixed point $\tau \in \partial\Delta$ generated by f, then the angular derivatives $F_t'(\tau)$ for all $t > 0$ are finite if and only if the angular derivatives $f'(\tau) =: \beta$ exists finitely. Moreover, in this case $F_t'(\tau) = e^{-\beta t}$ (see [19], [14], [13]).

As far as the higher order angular derivatives are concerned, even for the Denjoy–Wolff point one cannot assert that they do exist. Consider, for example, the parabolic holomorphic self-mapping F of Δ defined by

$$F(z) := \frac{2z + (1-z)\mathrm{Log}\left(\frac{2}{1-z}\right)}{2 + (1-z)\mathrm{Log}\left(\frac{2}{1-z}\right)}, \quad z \in \Delta,$$

where Log is the principal branch of the logarithm (see ([9])). The Denjoy–Wolff point of this mapping is $\tau = 1$. Consequently, there exists $\angle \lim\limits_{z\to 1} \frac{\partial F(z)}{\partial z}$. However, the angular limit $\angle \lim\limits_{z\to 1} \frac{\partial^2 F(z)}{\partial z^2}$ does not exist finitely.

In Theorem 1 below we show that the existence of the angular derivatives $f''(\tau)$ and $f'''(\tau)$ of the generator f of a semigroup $\{F_t\}_{t\geq 0}$ at a boundary fixed point τ implies that for each $t>0$, the angular derivatives $F_t''(\tau) := \angle\lim\limits_{z\to\tau}\frac{\partial^2 F(z)}{\partial z^2}$ and $F_t'''(\tau) := \angle\lim\limits_{z\to\tau}\frac{\partial^3 F(z)}{\partial z^3}$ also exist. Moreover, we give formulae which connect these derivatives. In the proof we use the following lemma.

Lemma 1 (see [17], p. 303) *Let $F\in \operatorname{Hol}(\Delta)$ and let $\tau\in\partial\Delta$ be a boundary fixed point of F. If F is conformal at τ, then nontangential convergence of z to τ implies that $F(z)$ converges to τ nontangentially.*

Theorem 1 *Let $S=\{F_t\}_{t\geq 0}$ be a one-parameter continuous semigroup generated by $f\in \operatorname{Hol}(\Delta,\mathbb{C})$ and let $\tau\in\partial\Delta$ be a boundary null point of f.*

(i) If $f'(\tau) := \angle\lim\limits_{z\to\tau} f'(\tau)$ exists finitely, then for each $t\geq 0$, $F_t'(\tau) := \angle\lim\limits_{z\to\tau} F'(z)$ also exists and

$$F_t'(\tau) = e^{-\beta t}, \tag{10}$$

where $\beta = f'(\tau)$.

(ii) If $f''(\tau) := \angle\lim\limits_{z\to\tau} f''(z)$ exists finitely, then for each $t\geq 0$, $F_t''(\tau) := \angle\lim\limits_{z\to\tau} F''(z)$ also exists and

$$F_t''(\tau) = \begin{cases} -\alpha t, \ \ \beta = 0 \\ \frac{\alpha}{\beta}e^{-\beta t}(e^{-\beta t}-1), \ \ \beta\neq 0, \end{cases} \tag{11}$$

where $\beta = f'(\tau)$, $\alpha = f''(\tau)$.

(iii) If $f'''(\tau) := \angle\lim\limits_{z\to\tau} f'''(z)$ exists finitely, then for each $t\geq 0$, $F_t'''(\tau) := \angle\lim\limits_{z\to\tau} F'''(z)$ also exists and

$$F_t'''(\tau) = \begin{cases} \frac{3}{2}\alpha^2 t^2 - \gamma t, \ \ \beta = 0 \\ \left(\frac{3\alpha^2}{2\beta^2}+\frac{\gamma}{2\beta}\right)e^{-3\beta t} - 3\frac{\alpha^2}{\beta^2}e^{-2\beta t} + \left(\frac{3\alpha^2}{2\beta^2}-\frac{\gamma}{2\beta}\right)e^{-\beta t}, \ \ \beta\neq 0, \end{cases} \tag{12}$$

where $\beta = f'(\tau)$, $\alpha = f''(\tau)$, $\gamma = f'''(\tau)$.

Proof. Since assertion (i) has been proved in [19] (see also [7] and [13]), we only present here proofs of assertions (ii) and (iii).

(ii) We have already mentioned above that semigroup elements solve the Cauchy problem (1). Differentiating the equality

$$\frac{\partial F_t(z)}{\partial t} + f(F_t(z)) = 0, \ z \in \Delta, \ t \geq 0, \tag{13}$$

two times with respect to $z \in \Delta$, we get

$$\frac{\partial}{\partial t}\left(\frac{\partial^2 F_t(z)}{\partial z^2}\right) + f''(F_t(z))\left(\frac{\partial F_t(z)}{\partial z}\right)^2 + f'(F_t(z))\frac{\partial^2 F_t(z)}{\partial z^2} = 0 \tag{14}$$

for all $z \in \Delta$ and $t \geq 0$.

Define the functions $p(z,t) := f'(F_t(z))$, $q(z,t) := -f''(F_t(z))\left(\frac{\partial F_t(z)}{\partial z}\right)^2$ and $u_2(z,t) := \frac{\partial^2 F_t(z)}{\partial z^2}$, $z \in \Delta$, $t \geq 0$. It is clear that $u_2(z,0) = 0$. Rewriting (14) in the form

$$\frac{\partial u_2(z,t)}{\partial t} + p(z,t)u_2(z,t) = q(z,t), \ z \in \Delta, \ t \geq 0,$$

we find

$$u_2(z,t) = e^{-\int_0^t p(z,s)ds} \cdot \int_0^t q(z,s)e^{\int_0^s p(z,\varsigma)d\varsigma}ds.$$

Now we fix t and let z tend to τ nontangentially in the right-hand side of this equality. Since $\angle\lim\limits_{z\to\tau} f''(z) := \alpha$ exists finitely, by Proposition 2, the angular limit $\angle\lim\limits_{z\to\tau} f'(z) := \beta$ also exists finitely. Consequently, for each $t \geq 0$, τ is a boundary fixed point of F_t and, by item (i), $\angle\lim\limits_{z\to\tau} F_t'(z) = e^{-\beta t} \neq 0$ (see Theorem 2 in [19]). Hence, by Lemma 1, $F_t(z)$ converges to τ nontangentially as z tends to τ nontangentially for each $t > 0$, and we can conclude that $\angle\lim\limits_{z\to\tau} p(z,t) = \beta$ and $\angle\lim\limits_{z\to\tau} q(z,t) = -\alpha e^{-2\beta t}$ for each $t > 0$. Hence,

$$\angle\lim_{z\to\tau}\left(e^{-\int_0^t p(z,t)ds} \cdot \int_0^t q(z,s)e^{\int_0^s p(z,\varsigma)d\varsigma}ds\right) =$$

$$= e^{-\int_0^t \angle\lim\limits_{z\to\tau} p(z,s)ds} \cdot \int_0^t \angle\lim_{z\to\tau} q(z,s) \cdot e^{\int_0^s \angle\lim\limits_{z\to\tau} p(z,\varsigma)d\varsigma}ds =$$

$$= -\alpha e^{-\beta t}\int_0^t e^{-\beta s}ds.$$

Therefore if $\beta = 0$, then

$$\angle \lim_{z\to\tau} \frac{\partial^2 F_t(z)}{\partial z^2} = -\alpha t, \;\; 0 \le t < \infty.$$

If $\beta \neq 0$, then

$$\angle \lim_{z\to\tau} \frac{\partial^2 F_t(z)}{\partial z^2} = \frac{\alpha}{\beta} e^{-\beta t} \cdot \left(e^{-\beta t} - 1\right).$$

(iii) Differentiating equality (13) three times with respect to $z \in \Delta$, we get

$$\frac{\partial}{\partial t}\left(\frac{\partial^3 F_t(z)}{\partial z^3}\right) + f'''(F_t(z))\left(\frac{\partial F_t(z)}{\partial z}\right)^3 + 3f''(F_t(z))\frac{\partial F_t(z)}{\partial z} \cdot \frac{\partial^2 F_t(z)}{\partial z^2} +$$

$$+f'(F_t(z))\frac{\partial^3 F_t(z)}{\partial z^3} = 0, \;\; t \ge 0, \;\; z \in \Delta. \tag{15}$$

Define the functions

$$r(z,t) := -f'''(F_t(z)) \cdot \left(\frac{\partial F_t(z)}{\partial z}\right)^3 - 3f''(F_t(z)) \cdot \frac{\partial F_t(z)}{\partial z} \cdot \frac{\partial^2 F_t(z)}{\partial z^2}$$

and $u_3(z,t) := \frac{\partial^3 F_t(z)}{\partial z^3}$, $z \in \Delta$, $t \ge 0$. It is clear that $u_3(z,0) = 0$. Rewriting (15) in the form

$$\frac{\partial u_3(t,z)}{\partial t} + p(z,t)u_3(z,t) = r(z,t), \;\; t \ge 0,$$

we find

$$u_3(z,t) = e^{-\int_0^t p(z,s)ds} \cdot \int_0^t r(z,s)e^{\int_0^s p(z,\varsigma)d\varsigma}ds.$$

Now we fix t and let z tend to τ nontangentially in the right-hand side of this equality.

Once again, by the continuity of $p(\cdot,t)$ and $r(\cdot,t)$ in $D_{\tau,\nu} \cup \{\tau\}$, $\nu \in (0, \frac{\pi}{2})$,

$$\angle \lim_{z\to\tau}\left(e^{-\int_0^t p(z,s)ds} \cdot \int_0^t q(z,s)e^{\int_0^s p(z,\varsigma)d\varsigma}ds\right) =$$

$$= e^{-\int_0^t \angle \lim_{z\to\tau} p(z,s)ds} \cdot \int_0^t \angle \lim_{z\to\tau} q(z,s) \cdot e^{\int_0^s \angle \lim_{z\to\tau} p(z,\varsigma)d\varsigma} ds =$$

$$= -e^{-\beta t} \cdot \int_0^t \left(\gamma e^{-3\beta s} + 3\alpha e^{-\beta s} \cdot \angle \lim_{z\to\tau} \frac{\partial^2 F_s(z)}{\partial z^2}\right) e^{\beta s} ds.$$

By Proposition 2, the limit $\angle\lim\limits_{z\to\tau}\frac{\partial^2 F_t(z)}{\partial z^2}$ exists and by item (ii) proved above, it is given by equality (11).

Hence, the limit $\angle\lim\limits_{z\to\tau}\frac{\partial^3 F_t(z)}{\partial z^3}$ exists and in the parabolic case ($\beta=0$) it equals

$$\angle\lim_{z\to\tau}\frac{\partial^3 F_t(z)}{\partial z^3}=-\int_0^t(\gamma-3\alpha^2 s)ds=\frac{3\alpha^2t^2}{2}-\gamma t.$$

In the hyperbolic case ($\beta\neq 0$) this limit also exists and

$$\angle\lim_{z\to\tau}\frac{\partial^3 F_t(z)}{\partial z^3}=-e^{-\beta t}\cdot\int_0^t\left(\left(\gamma+\frac{3\alpha^2}{\beta}\right)e^{-2\beta s}-\frac{3\alpha^2}{\beta}e^{-\beta s}\right)ds=$$

$$=\left(\frac{3\alpha^2}{2\beta^2}+\frac{\gamma}{2\beta}\right)e^{-3\beta t}-3\frac{\alpha^2}{\beta^2}e^{-2\beta t}+\left(\frac{3\alpha^2}{2\beta^2}-\frac{\gamma}{2\beta}\right)e^{-\beta t}.$$

■

Corollary 1 *Let $f\in\mathrm{Hol}(\Delta,\mathbb{C})$ be the generator of a parabolic semigroup $\{F_t\}_{t\geq 0}$ with the Denjoy–Wolff point $\tau\in\partial\Delta$. If $\angle\lim\limits_{z\to\tau}f''(\tau)=\angle\lim\limits_{z\to\tau}f'''(\tau)=0$, then $F_t=\mathrm{I}$ for all $t\geq 0$.*

Indeed, these conditions imply that $F_t'(\tau)=1,\ F_t''(\tau)=F_t'''(\tau)=0$ for all $t\geq 0$ and, by [12], we get $F_t=\mathrm{I}$.

Remark 3 As a matter of fact, repeating our proof and using Remark 1, one can show that the angular limits in Theorem 1 can be replaced by unrestricted limits. Namely:

Let $S=\{F_t\}_{t\geq 0}$ be the semigroup generated by f. Assume that for each $t>0$ the unrestricted limit $\lim\limits_{\substack{z\to\tau\\ z\in\Delta}}F(z)$ exists, where τ is a boundary null point of f. The following assertions hold:

(i) If the unrestricted limit $\beta:=\lim\limits_{\substack{z\to\tau\\ z\in\Delta}}f'(z)$ exists finitely, then

$$\lim_{\substack{z\to\tau\\ z\in\Delta}}F_t'(z)=e^{-\beta t}\quad \textit{for each } t\geq 0.$$

(ii) If the unrestricted limit $\alpha:=\lim\limits_{\substack{z\to\tau\\ z\in\Delta}}f''(z)$ exists finitely, then

$$\lim_{\substack{z\to\tau\\ z\in\Delta}}F_t''(z)=\begin{cases}-\alpha t, & \beta=0\\ \frac{\alpha}{\beta}e^{-\beta t}(e^{-\beta t}-1), & \beta\neq 0,\end{cases}\tag{16}$$

for each $t \geq 0$.

(iii) If the unrestricted limit $\gamma := \lim\limits_{\substack{z \to \tau \\ z \in \Delta}} f'''(z)$ exists finitely, then

$$\lim_{\substack{z \to \tau \\ z \in \Delta}} F_t'''(z) = \begin{cases} \frac{3}{2}\alpha^2 t^2 - \gamma t, & \beta = 0 \\ \left(\frac{3\alpha^2}{2\beta^2} + \frac{\gamma}{2\beta}\right) e^{-3\beta t} - 3\frac{\alpha^2}{\beta^2} e^{-2\beta t} + \left(\frac{3\alpha^2}{2\beta^2} - \frac{\gamma}{2\beta}\right) e^{-\beta t}, & \beta \neq 0, \end{cases} \tag{17}$$

for each $t \geq 0$.

Remark 4 *The arguments used in the proof of Theorem 1 can be used to derive analogous results for derivatives of any order $k \geq 4$.*

3 Semigroups with an interior fixed point

In our proofs we use the two following facts established by C. C. Cowen in [10].

Proposition 3 *Let F, G_1, G_2 be holomorphic self-mappings of Δ, not automorphisms of Δ, and let G_1 and G_2 commute with F. Suppose that $\tau \in \overline{\Delta}$ is the Denjoy–Wolff point of F and that $0 < |F'(\tau)| < 1$. Then G_1 and G_2 commute with each other.*

Proposition 4 *Let F and G be two commuting holomorphic self-mappings of Δ, not automorphisms of Δ, and let $\tau \in \overline{\Delta}$ be their common Denjoy–Wolff point.*

(i) If $F'(\tau) = 0$, then $G'(\tau) = 0$.
(ii) If $0 < |F'(\tau)| < 1$, then $0 < |G'(\tau)| < 1$.
(iii) If $F'(\tau) = 1$, then $G'(\tau) = 1$.

The following fact is more or less known (see, for example, [1]).

Proposition 5 *Let $S = \{F_t\}_{t \geq 0}$ be a semigroup in Δ. Assume F_{t_0} is an automorphism of Δ for some $t_0 > 0$; then each element F_t of S is an automorphism of Δ.*

We now begin our investigation of commuting semigroups. Note that in all the following theorems the condition $F_1 \circ G_1 = G_1 \circ F_1$ can be replaced by the condition $F_p \circ G_q = G_q \circ F_p$ for some $p, q > 0$.

Theorem 2 (dilation case) *Let $S_1 = \{F_t\}_{t\geq 0}$ and $S_2 = \{G_t\}_{t\geq 0}$ be two continuous semigroups on Δ generated by f and g, respectively, and let $F_1 \circ G_1 = G_1 \circ F_1$. Suppose that f has an interior null point $\tau \in \Delta$.*

(i) If S_1 and S_2 are not groups of automorphisms of Δ, then they commute.

(ii) If S_1 is a nontrivial group of elliptic automorphisms of Δ and S_2 is a semigroup of self-mappings of Δ, then S_1 and S_2 commute if and only if S_2 is a semigroup of linear fractional transformations of the form

$$G_t(z) = m_\tau(e^{-at} \cdot m_\tau(z)) \tag{18}$$

for some $a \in \mathbb{C}$, where $m_\tau(z) = \frac{\tau - z}{1-\overline{\tau}z}$.

Note that the function G_t defined by equality (18) is a self-mapping of Δ if and only if $\operatorname{Re} a \geq 0$.

Proof. Since τ is an interior null point of the generator f, it is the unique interior fixed point of the semigroup S_1 (see [1]). The commutativity of F_1 and G_1 implies that τ is a fixed point of G_1 and, consequently, τ is a fixed point of G_t for each $t > 0$.

(i) If S_1 and S_2 are not groups of automorphisms of Δ, then $0 < |F_t'(\tau)| < 1$ and $0 < |G_t'(\tau)| < 1$ for all $t > 0$, by the Schwarz–Pick Lemma and the univalence of F_t and G_t on Δ for all $t \geq 0$.

The function G_1 commutes with F_1 (by our assumption) and, for each $t \geq 0$, the mapping F_t commutes with F_1 (by the semigroups property). Therefore Proposition 3 implies that $G_1 \circ F_t = F_t \circ G_1$ for all $t \geq 0$.

Fix an arbitrary $t > 0$. Similarly, since $G_1 \circ F_t = F_t \circ G_1$ and $G_1 \circ G_s = G_s \circ G_1$ for all $s \geq 0$, we get, by Proposition 3, that $G_s \circ F_t = F_t \circ G_s$ for all $s \geq 0$. Hence, the semigroups S_1 and S_2 commute, as claimed.

(ii) Since S_1 is a group of elliptic automorphisms of Δ with a fixed point $\tau \in \Delta$, the functions F_t are of the form (see [3])

$$F_t(z) = m_\tau(e^{i\varphi t} m_\tau(z)) \text{ for some } \varphi \in \mathbb{R}.$$

Let $G_t(z) = m_\tau(e^{-at} m_\tau(z))$. Using the equality $m_\tau(m_\tau(z)) = z$, we get

$$F_t(G_s(z)) = m_\tau(e^{i\varphi t} m_\tau(m_\tau(e^{-as} m_\tau(z)))) = m_\tau(e^{i\varphi t} e^{-as} m_\tau(z)) =$$

$$= m_\tau(e^{-as} e^{i\varphi t} m_\tau(z)) = m_\tau(e^{-as} m_\tau(m_\tau(e^{i\varphi t} m_\tau(z)))) = G_s(F_t(z)).$$

Conversely, suppose that $F_t \circ G_s = G_s \circ F_t$ for all $s, t \geq 0$. Denote

$$\widetilde{F}_t(z) = e^{i\varphi t} z, \ \widetilde{G}_t = m_\tau \circ G_t \circ m_\tau.$$

Then $\{\widetilde{F}_t\}_{t \geq 0}$ is a group of automorphisms of Δ with a fixed point at zero, and $\{\widetilde{G}_t\}_{t \geq 0}$ is a semigroup of self-mappings of Δ with a fixed point at zero. It is obvious that the semigroups $\{\widetilde{F}_t\}_{t \geq 0}$ and $\{\widetilde{G}_t\}_{t \geq 0}$ commute. Consequently, their generators $\widetilde{g}(z)$ and $\widetilde{f}(z) = -i\varphi z$ are proportional (see [12]). So $\widetilde{g}(z) = az$ for some $a \in \mathbb{C}$. Therefore $\widetilde{G}(z) = e^{-at} z$ and $G_t(z) = m_\tau(e^{-at} m_\tau(z))$. ■

We see from this theorem that if S_1 is a group of elliptic automorphisms, the commutativity of F_1 and G_1 does not imply that the semigroups S_1 and S_2 commute. Nevertheless, in this case one can still obtain some additional information about the semigroup S_2. The following assertions explain our claim.

Proposition 6 *If $S_1 = \{F_t\}_{t \geq 0}$ is a group of elliptic automorphims whereas $S_2 = \{G_t\}_{t \geq 0}$ is a semigroup of self-mappings of Δ which are not automorphisms, then the commutativity of F_1 and G_1 implies that $F_1 \circ G_t = G_t \circ F_1$ for all $t \geq 0$.*

Proof. Let $\tau \in \Delta$ be the common fixed point of S_1 and S_2. Then the functions F_t are of the form $F_t(z) = m_\tau(e^{i\varphi t} m_\tau(z)),\ \varphi \in \mathbb{R},\ z \in \Delta$, where $m_\tau(z) = \frac{\tau - z}{1 - \overline{\tau} z}$.

Denote $\widetilde{F}_t(z) = e^{i\varphi t} z$ and $\widetilde{G}_t(z) = m_\tau(G_t(m_\tau(z)))$. Then $\{\widetilde{F}_t\}_{t \geq 0}$ is a group of automorphisms of Δ with its common fixed point at zero, and $\{\widetilde{G}_t\}_{t \geq 0}$ is a semigroup of self-mappings of Δ which are not automorphisms with its common fixed point also at zero.

It is obvious that for each $t > 0$, F_1 and G_t commute if and only if $\widetilde{F}_1$ and $\widetilde{G}_t$ commute. Hence, by our assumption, $\widetilde{F}_1 \circ \widetilde{G}_1 = \widetilde{G}_1 \circ \widetilde{F}_1$ or, which is the same, $e^{i\varphi} \widetilde{G}_1(z) = \widetilde{G}_1(e^{i\varphi} z)$. It follows that for all $n \in \mathbb{N}$, $\widetilde{F}_1 \circ \widetilde{G}_n = \widetilde{G}_n \circ \widetilde{F}_1$, where $\widetilde{G}_n$ are the iterates of $\widetilde{G}_1$, i.e., $\widetilde{G}_n = \widetilde{G}_1 \circ \widetilde{G}_{n-1}$.

Since $\widetilde{G}_1$ is a self-mapping of Δ (which is not an automorphism) with a fixed point at the origin, there exists a unique univalent solution h of the functional equation

$$h(\widetilde{G}_1(z)) = \alpha h(z), \ \text{with } \alpha = \widetilde{G}_1'(0),$$

normalized by $h(0) = 0,\ h'(0) = 1$ (see, for example, [18]). This solution is given by

$$h(z) = \lim_{n\to\infty} \frac{\widetilde{G}_n(z)}{\alpha^n}.$$

Moreover, by [11], for all real positive t,

$$h(\widetilde{G}_t(z)) = \alpha^t h(z)\,.$$

Therefore,

$$h(\widetilde{F}_1(\widetilde{G}_t(z))) = h(e^{i\varphi}\widetilde{G}_t(z)) = \lim_{n\to\infty} \frac{\widetilde{G}_n(e^{i\varphi}\widetilde{G}_t(z))}{\alpha^n} = \lim_{n\to\infty} \frac{e^{i\varphi}\widetilde{G}_n(\widetilde{G}_t(z))}{\alpha^n} =$$

$$= e^{i\varphi}h(\widetilde{G}_t(z)) = e^{i\varphi}\alpha^t h(z) = \alpha^t \lim_{n\to\infty} \frac{e^{i\varphi}\widetilde{G}_n(z)}{\alpha^n} = \alpha^t \lim_{n\to\infty} \frac{\widetilde{G}_n(e^{i\varphi}z)}{\alpha^n} =$$

$$= \alpha^t h(e^{i\varphi}z) = h(\widetilde{G}_t(e^{i\varphi}z)) = h(\widetilde{G}_t(\widetilde{F}_1(z)))$$

and, by the univalence of h, we get $\widetilde{F}_1 \circ \widetilde{G}_t = \widetilde{G}_t \circ \widetilde{F}_1$ for all $t \geq 0$. Consequently, F_1 and G_t commute for all $t \geq 0$. ■

Corollary 2 *Let $S_1 = \{F_t\}_{t\geq 0}$ be a group of elliptic automorphisms of Δ, i.e., $F_t(z) = m_\tau(e^{i\varphi t}m_\tau(z))$, $\varphi \in \mathbb{R}$, $\tau \in \Delta$, and let $S_2 = \{G_t\}_{t\geq 0}$ be a semigroup of self-mappings of Δ. Suppose that $\frac{\varphi}{\pi}$ is an irrational number and F_1 and G_1 commute. Then $G_t(z) = m_\tau(e^{-at}m_\tau(z))$, $a \in \mathbb{C}$, and, consequently, the semigroups S_1 and S_2 commute.*

Proof. Once again, we define the functions $\widetilde{F}_t = e^{i\varphi t}z$ and $\widetilde{G}_t = m_\tau \circ G_t \circ m_\tau$. The commutativity of F_1 and G_1 implies that $\widetilde{F}_1 \circ \widetilde{G}_1 = \widetilde{G}_1 \circ \widetilde{F}_1$ and, by Proposition 6, $\widetilde{F}_1 \circ \widetilde{G}_t = \widetilde{G}_t \circ \widetilde{F}_1$ for all $t \geq 0$. Therefore $\widetilde{G}_t(e^{in\varphi}z) = e^{in\varphi}\widetilde{G}_t(z)$ for all $n \in \mathbb{N}$. Since the set $\{e^{in\varphi}\}_{n\in\mathbb{N}}$ is dense in the unit circle, $\widetilde{G}_t(\lambda z) = \lambda\widetilde{G}_t(z)$ for all λ with $|\lambda| = 1$ and $z \in \Delta$, by the continuity of $\widetilde{G}_t$ on Δ.

Fix $0 \neq z \in \Delta$ and $t > 0$, and consider the analytic function $q(\lambda)$ on the closed unit disk defined by

$$q(\lambda) = \begin{cases} \dfrac{\widetilde{G}_t(\lambda z)}{\lambda}, \ \lambda \neq 0, \\ \\ \displaystyle\lim_{\lambda\to 0} \frac{\widetilde{G}_t(\lambda z)}{\lambda} = z\frac{\partial}{\partial w}\widetilde{G}_t(w)\Big|_{w=0}, \quad \lambda = 0. \end{cases} \tag{19}$$

This function is constant on the unit circle: $q(\lambda) = \widetilde{G}_t(z)$. Moreover, $q(\lambda) \neq 0$ for all $\lambda \in \Delta$. Therefore $q(\lambda) = \widetilde{G}_t(z)$ for all $\lambda \in \overline{\Delta}$. So for each $z \neq 0$ and $t > 0$, $\widetilde{G}_t(\lambda z) = \lambda \widetilde{G}_t(z)$. Consequently, this equality holds for all $z \in \Delta$. Hence $\widetilde{G}_t$ is a linear function for each $t > 0$, i.e., $\widetilde{G}_t(z) = e^{-at} z$ for some $a \in \mathbb{C}$, $\operatorname{Re} a \geq 0$, and the assertion follows. ■

In contrast with this corollary, if $\frac{\varphi}{\pi}$ is a rational number, the semigroups S_1 and S_2 do not necessarily commute. The following example gives a large class of semigroups $S_2 = \{G_t\}_{t \geq 0}$ such that $F_1 \circ G_t = G_t \circ F_1$ for all $t \geq 0$, but the semigroups S_1 and S_2 do not commute.

Example. Let $S_1 = \{F_t\}_{t \geq 0}$, where $F_t(z) = e^{i \frac{2\pi}{n} t} z$, $n \in \mathbb{N}$, and let $S_2 = \{G_t\}_{t \geq 0}$ be the semigroup generated by $g(z) = zp(z^n)$, where $\operatorname{Re} p(z^n) \geq 0$ for all $z \in \Delta$. Then $F_1 \circ G_t = G_t \circ F_1$ for all $t \geq 0$.

Indeed, denote $u = u(t, z) := G_t(z)$. Then u is the unique solution of the Cauchy problem

$$\begin{cases} \dfrac{\partial u}{\partial t} + up(u^n) = 0, \\ u(0, z) = z, \quad z \in \Delta, \end{cases} \tag{20}$$

and, consequently,

$$\int_z^{G_t(z)} \frac{d\varsigma}{\varsigma p(\varsigma^n)} = -t \quad \text{for all} \quad z \in \Delta. \tag{21}$$

Substituting $e^{i \frac{2\pi}{n}} z$ instead of z, we get

$$\int_{e^{i \frac{2\pi}{n}} z}^{G_t(e^{i \frac{2\pi}{n}} z)} \frac{d\varsigma}{\varsigma p(\varsigma^n)} = -t.$$

Now substitute $\varsigma = e^{i \frac{2\pi}{n}} w$:

$$\int_z^{G_t(e^{i \frac{2\pi}{n}} z) e^{-i \frac{2\pi}{n}}} \frac{dw}{wp(w^n e^{i 2\pi})} = \int_z^{G_t(e^{i \frac{2\pi}{n}} z) e^{-i \frac{2\pi}{n}}} \frac{dw}{wp(w^n)} = -t, \; z \in \Delta. \tag{22}$$

Equalities (21) and (22) imply that

$$\int_{G_t(z)}^{G_t(e^{i \frac{2\pi}{n}} z) e^{-i \frac{2\pi}{n}}} \frac{dw}{wp(w^n)} = 0, \; z \in \Delta. \tag{23}$$

By the uniqueness of the solution to the Cauchy problem (20), the equation

$$\int_z^u \frac{dw}{wp(w^n)} = -s,\ \ s \geq 0,\ \ z \in \Delta,$$

has the unique solution $u = G_s(z)$ for each $s \geq 0$. Thus, it follows from (23) that $G_t(e^{i\frac{2\pi}{n}}z)e^{-i\frac{2\pi}{n}} = G_0(G_t(z)) = G_t(z)$. Hence, $G_t(e^{i\frac{2\pi}{n}}z) = e^{i\frac{2\pi}{n}}G_t(z)$. Therefore F_1 commutes with G_t for all $t \geq 0$. At the same time, if p is not a constant function, the semigroups do not commute because their generators are not proportional.

4 Semigroups of hyperbolic type

We start this section with an assertion which is of independent interest.

Proposition 7 *Let F and G be two commuting holomorphic self-mappings of Δ and assume that G is not the identity. If F is of hyperbolic type, then G is of hyperbolic type too.*

Proof. If F is a hyperbolic automorphism of Δ, then by Lemma 2.1 in [15] G is a hyperbolic automorphism of Δ.

Let F be a holomorphic self-mapping of Δ which is not an automorphism of Δ. In this case, by a result in [2], the mappings F and G have a common Denjoy–Wolff point $\tau \in \partial\Delta$. We have to show that G is of hyperbolic type, i.e., $0 < G'(\tau) < 1$. Suppose, to the contrary, that G is of parabolic type, i.e., $G'(\tau) = 1$. Then, by Proposition 4(ii), G must be a parabolic automorphism.

Denote $g := C \circ G \circ C^{-1}$ and $f := C \circ F \circ C^{-1}$, where $C(z) = \frac{\tau+z}{\tau-z}$. Then f and g are two commuting holomorphic self-mappings of the right half-plane $\mathbb{H} = \{z : \mathrm{Re} z > 0\}$ with their common Denjoy–Wolff point at infinity. Moreover, g is a parabolic automorphism of $\mathbb{H}$ while f is a hyperbolic self-mapping of $\mathbb{H}$. Consequently, f and g are of the forms (see [18]):

$$f(w) = cw + \Gamma_F(w) \quad \text{with} \quad c = \frac{1}{F'(\tau)} > 1 \quad \text{and} \quad \angle \lim_{w\to\infty} \frac{\Gamma_F(w)}{w} = 0,$$

and

$$g(w) = w + ib \quad \text{with} \quad b \in \mathbb{R} \setminus \{0\} \quad \text{and} \quad w \in \mathbb{H}.$$

By a simple calculation and the commutativity of f and g_n, we infer from the above representations that

$$f(w+nib) = f(w) + nib, \quad w \in \mathbb{H}. \tag{24}$$

Hence,

$$\frac{f(w+nib)}{w+nib} = \frac{f(w)}{w+nib} + \frac{nib}{w+nib}, \quad w \in \mathbb{H}.$$

Letting $n \to \infty$, we obtain that for each $w \in \mathbb{H}$, the limit $\lim\limits_{n\to\infty} \frac{f(w+nib)}{w+nib}$ exists and equals 1.

Fix $w_0 \in \mathbb{H}$. Consider the curve $l := \{w_0 + it : t \in \mathbb{R},\ \operatorname{sgn} t = \operatorname{sgn} b\}$. We intend to show that the limit $\lim\limits_{l\ni z\to\infty} \frac{f(z)}{z}$ exists and equals 1.

To this end, fix an arbitrary $\varepsilon > 0$ and take $N \in \mathbb{N}$ such that

$$N > \frac{1}{|b|}\left(\frac{|f(w)-w|}{\varepsilon} + |w|\right) \quad \text{and} \quad N > \frac{|w|}{|b|}$$

for all $w \in [w_0, w_0 + ib]$.

Then $\left|\frac{f(z)}{z} - 1\right| < \varepsilon$ for all $z \in l$ with $\operatorname{sgn} b \cdot \operatorname{Im} z > \operatorname{sgn} b(\operatorname{Im} w_0 + Nb)$.

Indeed, if $\operatorname{sgn} b \cdot \operatorname{Im} z > \operatorname{sgn} b(\operatorname{Im} w_0 + Nb)$, then $z = \alpha + ikb$ for some $\alpha \in [w_0, w_0 + ib]$ and $k \geq N$.

Hence, $k|b| \geq |\alpha|$ and $k > \frac{1}{|b|}\left(\frac{|f(\alpha)-\alpha|}{\varepsilon} + |\alpha|\right)$. Consequently, $|\alpha + ikb| > k|b| - |\alpha| > \frac{|f(\alpha)-\alpha|}{\varepsilon}$.

Now using (24), we obtain that

$$\left|\frac{f(z)}{z} - 1\right| = \left|\frac{f(\alpha+kib)}{\alpha+kib} - 1\right| = \left|\frac{f(\alpha)-\alpha}{\alpha+ikb}\right| < \varepsilon.$$

Thus $\lim\limits_{l\ni z\to\infty} \frac{f(z)}{z} = 1$. It now follows from Lindelöf's theorem (see, for example, [20]) that $\angle \lim\limits_{z\to\infty} \frac{f(z)}{z} = 1$, which contradicts our assumption. Therefore the mapping G is indeed of hyperbolic type. ■

Theorem 3 (hyperbolic case) *Let $S_1 = \{F_t\}_{t\geq 0}$ and $S_2 = \{G_t\}_{t\geq 0}$ be continuous semigroups on Δ generated by f and g, respectively, and assume that $F_1 \circ G_1 = G_1 \circ F_1$. Suppose that f has a boundary null point $\tau \in \partial\Delta$, such that $f'(\tau) := \angle \lim\limits_{z\to\tau} f'(z) > 0$, i.e., the semigroup S_1 is of hyperbolic type. Then the semigroups S_1 and S_2 commute. Thus, if $g \neq 0$ then S_2 is also of hyperbolic type.*

Proof. By our assumption, τ is the Denjoy–Wolff point of the semigroup S_1.

First we suppose that S_1 and S_2 consist of automorphisms of Δ. Since $f'(\tau) > 0$, S_1 consists of hyperbolic automorphisms of Δ and its generator f is of the form

$$f(z) = \frac{a_1}{\tau - \varsigma}(z - \tau)(z - \varsigma),$$

where a_1 is a positive real number and ς is the second common fixed point of the semigroup S_1 (see [3]).

The commutativity of F_1 and G_1 implies that G_1 has the same fixed points τ and ς; consequently, S_2 consists of hyperbolic automorphisms of Δ, and its generator g is of the form

$$g(z) = \frac{a_2}{\varsigma - \tau}(z - \tau)(z - \varsigma),$$

where a_2 is a non-zero real number. Hence, $g(z) = -\frac{a_2}{a_1} f(z)$, and by Theorem 3 in [12], the semigroups commute.

Suppose now that at least one of the semigroups S_1 and S_2 consists of self-mappings of Δ which are not automorphisms. By a result in [2], τ is the common Denjoy–Wolff point of S_1 and S_2. Moreover, by Theorem 1, $\alpha := F_1'(\tau) = e^{-f'(\tau)} \in (0, 1)$. Consequently, by Proposition 7, $\beta := G_1'(\tau) \in (0, 1)$.

Since F_1 is a hyperbolic self-mapping of Δ, the limit (where $F_n = F_1^n$ is the n-th iterate of F_1)

$$h(z) := \lim_{n\to\infty} \frac{1 - F_n(z)}{1 - F_n(0)}\ , \quad z \in \Delta,$$

exists and is not constant (see [14]). Moreover, for each $t > 0$, the function h is the unique univalent solution of Schröder's functional equation

$$h(F_t(z)) = \alpha^t h(z)$$

normalized by $h(0) = 1$ (see [14] and [11]). Hence,

$$h(G_1(z)) = \lim_{n\to\infty} \frac{1 - F_n(G_1(z))}{1 - F_n(0)} = \lim_{n\to\infty} \frac{1 - G_1(F_n(z))}{1 - F_n(z)} \cdot \frac{1 - F_n(z)}{1 - F_n(0)} = \beta h(z).$$

Therefore

$$h(G_1(F_t(z))) = \beta h(F_t(z)) = \beta\alpha^t h(z) = \alpha^t h(G_1(z)) = h(F_t(G_1(z)))$$

for all $t \geq 0$ and $z \in \Delta$, and by the univalence of h, G_1 commutes with F_t for each $t \geq 0$.

Fix $t > 0$, and denote by σ the Kœnigs function for S_2:

$$\sigma(z) := \lim_{n\to\infty} \frac{1 - G_n(z)}{1 - G_n(0)}, \; z \in \Delta.$$

Since the mapping G_1 is of hyperbolic type, this limit exists and for each $s > 0$, the function σ is the unique univalent solution of Schröder's functional equation

$$\sigma(G_s(z)) = \beta^s \sigma(z)$$

normalized by $\sigma(0) = 1$. Hence,

$$\sigma(F_t(z)) = \lim_{n\to\infty} \frac{1 - G_n(F_t(z))}{1 - G_n(0)} = \lim_{n\to\infty} \frac{1 - F_t(G_n(z))}{1 - G_n(z)} \cdot \frac{1 - G_n(z)}{1 - G_n(0)} = \alpha^t \sigma(z).$$

Consequently,

$$\sigma(F_t(G_s(z))) = \alpha^t \sigma(G_s(z)) = \alpha^t \beta^s \sigma(z) = \beta^s \sigma(F_t(z)) = \sigma(G_s(F_t(z)))$$

for all $s > 0$ and $z \in \Delta$, and by the univalence of σ the semigroups commute. $\blacksquare$

5 Semigroups of parabolic type

For each $n = 0, 1, \ldots,$ we denote by $C^n_A(\tau)$, $\tau \in \Delta$, the class of functions $F \in \mathrm{Hol}(\Delta, \mathbb{C})$ which admit the representation

$$F(z) = \sum_{k=0}^{n} a_k (z - \tau)^k + \gamma(z), \tag{25}$$

where $\gamma \in \mathrm{Hol}(\Delta, \mathbb{C})$ and $\angle \lim\limits_{z\to\tau} \frac{\gamma(z)}{(z-\tau)^n} = 0$; and we say that $F \in C^n(\tau)$ when this expansion holds as $z \to \tau$ unrestrictedly.

To proceed we need the following auxiliary result.

Lemma 2 *Let $F, G \in \mathrm{Hol}(\Delta)$ be two commuting univalent parabolic mappings and let $\tau = 1$ be the Denjoy–Wolff point of F. If one of the following conditions*

(i) $F, G \in C^2(1)$, $F''(1) \neq 0$, $G''(1) \neq 0$;

(ii) $F, G \in C^2_A(1)$, $G''(1) \neq 0$, $\operatorname{Re} F''(1) > 0$;

(iii) $F, G \in C^3(1)$, $F''(1) = G''(1) = 0$, $F'''(1) \neq 0$, $G'''(1) \neq 0$

holds, then there exists a univalent function $\sigma \in \mathrm{Hol}(\Delta, \mathbb{C})$ *such that*

$$\sigma \circ F = \sigma + 1 \tag{26}$$

and

$$\sigma \circ G = \sigma + \lambda \quad \textit{with} \quad \lambda \in \mathbb{C}, \;\; \lambda \neq 0. \tag{27}$$

Proof. Consider $z_n^0 := F_n(0)$ and $\sigma_n(z) := \dfrac{F_n(z) - z_n^0}{z_{n+1}^0 - z_n^0}$, $\quad z \in \Delta$. Then $\sigma_n \in \mathrm{Hol}(\Delta, \mathbb{C})$ and the sequence $\{\sigma_n\}_{n=1}^{\infty}$ converges in the compact-open topology to a certain holomorphic map $\sigma \in \mathrm{Hol}(\Delta, \mathbb{C})$ such that (26) holds (by Theorem 2.1 in [8]). Since F is univalent in Δ, the solution σ of Abel's equation (26) is also univalent in Δ.

Now we show that σ satisfies (27). Denote $f = C \circ F \circ C^{-1}$, $g = C \circ G \circ C^{-1}$, $f, g \in \mathrm{Hol}(\mathbb{H}, \mathbb{H})$, where $\mathbb{H} = \{z : \operatorname{Re}(z) > 0\}$ and C is the Cayley transformation given by $C(z) = \frac{1+z}{1-z}$. Then f and g are commuting parabolic maps in $\mathrm{Hol}(\mathbb{H}, \mathbb{H})$ having ∞ as their common Denjoy–Wolff point.

Denote $w_0 := C(0) = 1$, $w_n^0 := f_n(1) = C(z_n^0)$,

$$w_n := f_n(w), \quad w_n \in \mathbb{H},$$

and

$$h_n(w) := \frac{w_n - w_n^0}{w_{n+1}^0 - w_n^0}\,, \qquad w \in \mathbb{H}.$$

Then $h_n \in \mathrm{Hol}(\mathbb{H}, \mathbb{C})$ and the sequence $\{h_n\}_{n=1}^{\infty}$ converges in the compact open topology to a holomorphic function $h \in \mathrm{Hol}(\mathbb{H}, \mathbb{C})$ such that $h \circ f = h + 1$ and $\sigma = h \circ C$ (see [8]).

Suppose that (i) holds. Then the following expansions of f and g at ∞ are satisfied (see [5]):

$$f(w) = w + F''(1) + \gamma_f(w), \;\; \lim_{w \to \infty} \gamma_f(w) = 0 \tag{28}$$

and

$$g(w) = w + G''(1) + \gamma_g(w), \;\; \lim_{w \to \infty} \gamma_g(w) = 0. \tag{29}$$

Hence,

$$h_n(g(w)) = \frac{f_n(g(w)) - w_n^0}{w_{n+1}^0 - w_n^0} = \frac{g(f_n(w)) - w_n^0}{w_{n+1}^0 - w_n^0}$$

$$= \frac{w_n + G''(1) + \gamma_g(w_n) - w_n^0}{w_{n+1}^0 - w_n^0} = \frac{w_n - w_n^0}{w_{n+1}^0 - w_n^0} + \frac{G''(1) + \gamma_g(w_n)}{w_{n+1}^0 - w_n^0}$$
$$= h_n(w) + \frac{G''(1) + \gamma_g(w_n)}{F''(1) + \gamma_f(w_n)} \cdot \frac{F''(1) + \gamma_f(w_n)}{w_{n+1}^0 - w_n^0}.$$

Letting $n \to \infty$, we obtain

$$h(g(w)) - h(w) = \frac{G''(1)}{F''(1)} \cdot \lim_{n\to\infty} \frac{F''(1) + \gamma_f(w_n)}{w_{n+1}^0 - w_n^0}. \tag{30}$$

Repeating this calculation with f instead of g, we find that

$$h(f(w)) = h(w) + \lim_{n\to\infty} \frac{F''(1) + \gamma_f(w_n)}{w_{n+1}^0 - w_n^0}.$$

At the same time, $h \circ f = h + 1$. Hence $\lim\limits_{n\to\infty} \dfrac{F''(1) + \gamma_f(w_n)}{w_{n+1}^0 - w_n^0} = 1$.

Rewrite (30) as follows:

$$h(g(w)) - h(w) = \lambda, \quad \text{where} \quad \lambda = \frac{G''(1)}{F''(1)} \neq 0 \quad \text{and} \quad w \in \mathbb{H}.$$

Substituting $h = \sigma \circ C^{-1}$ and $g = C \circ G \circ C^{-1}$ in the last equality we get (27).

If (ii) holds, then Theorem 14 in [9] implies that for each $z \in \Delta$, the sequence $\{F_n(z)\}_{n=1}^{\infty}$ converges to 1 (and, consequently, $\{w_n\}$ converges to ∞) nontangentially. So, in this case, one can repeat the proof of item (i), replacing the unrestricted limits in (28) and (29) by the angular limits.

Suppose now that (iii) holds. Then the following expansions of f and g at ∞ hold (see [5]):

$$f(w) = w - \frac{2}{3}\frac{F'''(1)}{w+1} + \Gamma_f(w), \ \lim_{w\to\infty} \Gamma_f(w)w = 0 \tag{31}$$

and

$$g(w) = w - \frac{2}{3}\frac{G'''(1)}{w+1} + \Gamma_g(w), \ \lim_{w\to\infty} \Gamma_g(w)w = 0. \tag{32}$$

Therefore

$$h_n(f(w)) = \frac{f(w_n) - w_n^0}{w_{n+1}^0 - w_n^0} = \frac{w_n - w_n^0}{w_{n+1}^0 - w_n^0} + \frac{-\frac{2}{3}\frac{F'''(1)}{w_n+1} + \Gamma_f(w_n)}{w_{n+1}^0 - w_n^0}$$

$$= h_n(w) + \frac{-\frac{2}{3}\frac{F'''(1)}{w_n+1} + \Gamma_f(w_n)}{w_{n+1}^0 - w_n^0}.$$

Letting $n \to \infty$, we obtain

$$h(f(w)) = h(w) + \lim_{n\to\infty} \frac{-\frac{2}{3}\frac{F'''(1)}{w_n+1} + \Gamma_f(w_n)}{w_{n+1}^0 - w_n^0}.$$

On the other hand, $h(f(w)) = h(w) + 1$. Hence,

$$\lim_{n\to\infty} \frac{-\frac{2}{3}\frac{F'''(1)}{w_n+1} + \Gamma_f(w_n)}{w_{n+1}^0 - w_n^0} = 1. \tag{33}$$

Now using (32), we find

$$h_n(g(w)) = \frac{g(w_n) - w_n^0}{w_{n+1}^0 - w_n^0} = \frac{w_n - \frac{2}{3}\frac{G'''(1)}{w_n+1} + \Gamma_g(w_n) - w_n^0}{w_{n+1}^0 - w_n^0}$$

$$= h_n(w) + \frac{-\frac{2}{3}G'''(1) + \Gamma_g(w_n)(w_n+1)}{-\frac{2}{3}F'''(1) + \Gamma_f(w_n)(w_n+1)} \cdot \frac{-\frac{2}{3}\frac{F'''(1)}{w_n+1} + \Gamma_f(w_n)}{w_{n+1}^0 - w_n^0}.$$

Letting $n \to \infty$ and using (33), we get

$$h(g(w)) - h(w) = \lambda,\ w \in \mathbb{H},\ \text{where } \lambda = \frac{G'''(1)}{F'''(1)} \neq 0.$$

Consequently, $\sigma \circ G - \sigma = \lambda$. ■

Following [8], we say that the function σ mentioned in the lemma is *the Kœnigs intertwining function* associated with F with respect to $z_0 = 0$.

Remark 5 *The function σ in Lemma 2 is completely determined by the function F. It does not depend on G. So if the conditions of the lemma hold for the same function F and another function $G_1 \in \mathrm{Hol}(\Delta)$, then we have the equality*

$$\sigma \circ G_1 = \sigma + \lambda_1$$

with the same function σ and a constant $\lambda_1 \neq 0$.

Theorem 4 (parabolic case) *Let $S_1 = \{F_t\}_{t\geq 0}$ and $S_2 = \{G_t\}_{t\geq 0}$ be two continuous semigroups on Δ generated by f and g, respectively, and let $F_1 \circ G_1 = G_1 \circ F_1$.*

Suppose that $\tau = 1$ is the boundary null point of f such that $f'(1) = 0$. If $S_1, S_2 \subset C^0(1)$ and one of the following conditions

(i) $f, g \in C^2(1)$, $f''(1) \neq 0$, $g''(1) \neq 0$;

(ii) $f, g \in C^3(1)$, $f''(1) = g''(1) = 0$

holds, then the semigroups commute.

Proof. Since τ is a boundary null point of f and $f'(\tau) = 0$, it is the common Denjoy–Wolff point of the semigroup S_1. The commutativity of F_1 and G_1 implies that τ is the Denjoy–Wolff point of G_1 (see [2]) and, consequently, τ is also the common Denjoy–Wolff point of the semigroup S_2 .

If (ii) holds and, in addition, either $f'''(1) = 0$ or $g'''(1) = 0$, then by Corollary 1 we have that either $F_t \equiv I$ or $G_t \equiv I$, respectively, and therefore the semigroups commute. Suppose that $f'''(1) \neq 0$ and $g'''(1) \neq 0$ in (ii). Then by Remark 3 above, one can replace conditions (i) and (ii) by

(i') $F_t, G_t \in C^2(1)$, $F_t''(1) \neq 0$, $G_t''(1) \neq 0$ for all $t > 0$;

(ii') $F_t, G_t \in C^3(1)$, $F_t''(1) = G_t''(1) = 0$, $F_t'''(1) \neq 0$, $G_t'''(1) \neq 0$, $t > 0$.

By our assumption, $F_1 \circ G_1 = G_1 \circ F_1$. Moreover, $F_1 \circ F_t = F_t \circ F_1$ for all $t \geq 0$. Therefore Lemma 2 implies that there exists the Kœnigs intertwining map σ for F_1 with respect to $z_0 = 0$, which satisfies

$$\sigma(F_1(z)) = \sigma(z) + 1, \ z \in \Delta, \tag{34}$$

$$\sigma(G_1(z)) - \sigma(z) = \lambda, \ z \in \Delta, \ \text{ for some } \lambda \neq 0, \tag{35}$$

and

$$\sigma(F_t(z)) - \sigma(z) = \beta(t), \ t > 0, \ z \in \Delta, \tag{36}$$

where $\beta(t) \neq 0$ for all $t > 0$.

Furthermore, $F_1 \circ G_1 = G_1 \circ F_1$ and $G_1 \circ G_s = G_s \circ G_1$ for all $s \geq 0$. Hence, by Lemma 2, there exists the Kœnigs intertwining map $\widetilde{\sigma}$ for G_1 with respect to $z_0 = 0$, which satisfies

$$\widetilde{\sigma}(G_1(z)) = \widetilde{\sigma}(z) + 1, \ z \in \Delta, \tag{37}$$

$$\widetilde{\sigma}(F_1(z)) - \widetilde{\sigma}(z) = \widetilde{\lambda}, \ z \in \Delta, \text{ for some } \widetilde{\lambda} \neq 0, \tag{38}$$

and

$$\widetilde{\sigma}(G_s(z)) - \widetilde{\sigma}(z) = \widetilde{\beta}(s),\ \ s > 0,\ \ z \in \Delta, \tag{39}$$

where $\widetilde{\beta}(s) \neq 0$ for all $s > 0$.

Assume that at least one of the mappings F_1, G_1 (for example, G_1) is of nonautomorphic type. (Note that if (ii') holds then for each $t > 0$, G_t and F_t are of nonautomorphic type by Theorem 4.4 in [18].) It follows from (38) and (39) that

$$\frac{\widetilde{\beta}(s)}{\widetilde{\lambda}}(\widetilde{\sigma}(F_1(z)) - \widetilde{\sigma}(z)) = \widetilde{\sigma}(G_s(z)) - \widetilde{\sigma}(z). \tag{40}$$

Rewrite (35) in the form

$$\frac{1}{\lambda}\sigma(G_1(z)) = \frac{1}{\lambda}\sigma(z) + 1. \tag{41}$$

By Theorem 3.1 in [8], equalities (37) and (41) imply that $\frac{1}{\lambda}\sigma = \widetilde{\sigma} + const.$, and so (40) is equivalent to

$$\frac{\widetilde{\beta}(s)}{\widetilde{\lambda}}(\sigma(F_1(z)) - \sigma(z)) = \sigma(G_s(z)) - \sigma(z) \tag{42}$$

or, by (34),

$$\frac{\widetilde{\beta}(s)}{\widetilde{\lambda}} = \sigma(G_s(z)) - \sigma(z). \tag{43}$$

Since the right-hand sides in (36) and (43) are differentiable in t and s, respectively, β and $\widetilde{\beta}$ are differentiable too. Hence,

$$\beta'(t) = \sigma'(F_t(z)) \cdot \frac{\partial F_t(z)}{\partial t} \ \text{ and } \ \frac{\widetilde{\beta}'(s)}{\widetilde{\lambda}} = \sigma'(G_s(z)) \cdot \frac{\partial G_s(z)}{\partial s}.$$

Letting $t \to 0^+$ and $s \to 0^+$ in these equalities, we obtain

$$\beta'(0) = -\sigma'(z) \cdot f(z) \ \text{ and } \ \frac{\widetilde{\beta}'(0)}{\widetilde{\lambda}} = -\sigma'(z) \cdot g(z),$$

where f and g are generators of the semigroups $\{F_t\}_{t\geq 0}$ and $\{G_t\}_{t\geq 0}$, respectively.

Since σ is univalent on Δ, the derivative $\sigma'(z) \neq 0$ for all $z \in \Delta$. Moreover, because the common Denjoy–Wolff point of S_1 and S_2 belongs to the boundary $\partial\Delta$, the generators f and g do not vanish on Δ. Therefore

$$f(z) = ag(z), \text{ where } a = \frac{\widetilde{\lambda}\beta'(0)}{\widetilde{\beta}'(0)},$$

and by [12], the semigroups commute.

Now let the mappings F_1 and G_1 be both of automorphic type. Note that in this case $F_1''(1)$ and $G_1''(1)$ cannot be zero and so we assume that (i') holds.

We have already seen in the proof of Lemma 2 that

$$\sigma(G_1(z)) - \sigma(z) = \frac{G_1''(1)}{F_1''(1)}. \tag{44}$$

Since $\operatorname{Re} F_1''(1) = 0$ and $\operatorname{Re} G_1''(1) = 0$ (see Theorem 4.4 in [5]), it follows that $\frac{G_1''(1)}{F_1''(1)} \in \mathbb{R} \setminus \{0\}$. Moreover, by Theorem 1,

$$F_t''(1) = -\alpha t \quad \text{and} \quad G_t''(1) = -\widetilde{\alpha} t, \ t > 0,$$

where $\alpha = f''(1) \neq 0$ and $\widetilde{\alpha} = g''(1) \neq 0$. So equality (44) has the form

$$\sigma(G_1(z)) - \sigma(z) = p, \quad \text{where} \quad p := \frac{\widetilde{\alpha}}{\alpha}. \tag{45}$$

On the other hand,

$$\sigma(F_t(z)) - \sigma(z) = \frac{F_t''(1)}{F_1''(1)} = \frac{\alpha t}{\alpha} = t \quad \text{for all} \quad t \geq 0. \tag{46}$$

First we suppose that $p > 0$. From (45) and (46) we have $\sigma(G_1(z)) = \sigma(F_p(z))$, $z \in \Delta$, and by the univalence of σ on Δ, $G_1(z) = F_p(z)$ for all $z \in \Delta$. Hence, $G_1 \circ F_t = F_t \circ G_1$ for all $t \geq 0$.

Fix $t > 0$ and repeat these considerations with G_1, F_t, G_s and $\widetilde{\sigma}$ instead of F_1, G_1, F_t and σ, respectively. Namely,

$$\widetilde{\sigma}(F_t(z)) - \widetilde{\sigma}(z) = \frac{F_t''(1)}{G_1''(1)} = \frac{\alpha t}{\widetilde{\alpha}} > 0$$

and

$$\widetilde{\sigma}(G_s(z)) - \widetilde{\sigma}(z) = \frac{G_s''(1)}{G_1''(1)} = s \quad \text{for all} \quad s > 0.$$

Denote $\widetilde{s} := \frac{\alpha t}{\widetilde{\alpha}} > 0$. Then $\widetilde{\sigma}(F_t(z)) = \widetilde{\sigma}(G_{\widetilde{s}}(z)),\ z \in \Delta$. By the univalence of $\widetilde{\sigma}$ on Δ we have $F_t(z) = G_{\widetilde{s}}(z)$. Therefore $G_s \circ F_t = F_t \circ G_s$ for all $s > 0$. Since $t > 0$ is arbitrary, it follows that the semigroups $S_1 = \{F_t\}_{t \geq 0}$ and $S_2 = \{G_s\}_{s \geq 0}$ commute.

Let now $p < 0$. Then by (46), $\sigma(F_{-p}(z)) - \sigma(z) = -p$ for all $z \in \Delta$. Hence, by (45),

$$\sigma(F_{-p}(G_1(z))) - \sigma(G_1(z)) = \sigma(z) - \sigma(G_1(z)), \quad z \in \Delta,$$

and, therefore,

$$\sigma(F_{-p}(G_1(z))) = \sigma(z), \quad z \in \Delta.$$

By the univalence of σ on Δ, $F_{-p}(G_1(z)) = z$. Consequently, $F_{-p} = G_1^{-1}$ on $G_1(\Delta)$. Since $F_{-p} \in \mathrm{Hol}(\Delta)$, G_1^{-1} is well defined on Δ and so G_1, as well as F_{-p}, are an automorphisms of Δ. Therefore, by Proposition 5, $\{F_t\}_{t \geq 0}$ is a semigroup of automorphisms. Consequently, it can be extended to a group $S_F = \{F_t\}_{t \in \mathbb{R}}$ and $G_1 = F_p^{-1} = F_{-p} \in S_F$. In particular, $G_1 \circ F_t = F_t \circ G_1$ for all $t \geq 0$.

Fix $t > 0$. In a similar way, using the commutativity of F_t and G_1, one can show that the semigroup $\{G_s\}_{s \geq 0}$ can be extended to a group $S_G = \{G_s\}_{s \in \mathbb{R}}$ and that $F_t \circ G_s = G_s \circ F_t$ for all $s, t \in \mathbb{R}$. ■

Remark 6 Note in passing that the proof of Theorem 4 implies the following interesting fact:

Let $S_1 = \{F_t\}_{t \geq 0}$ be a continuous semigroup of parabolic type on Δ generated by f with the Denjoy–Wolff point $\tau = 1$, and let G be a holomorphic self-mapping of Δ such that $F_1 \circ G = G \circ F_1$. If $f, G \in C^2(1)$ and $S_1 \subset C^0(1)$, then the condition $f''(1) \cdot G''(1) > 0$ implies that S_1 can be extended to a group of parabolic automorphisms of Δ and $G \in S_1$, hence G commute with all elements $F_t,\ t > 0$.

Remark 7 Note also that if in the assumptions of Theorem 4, $S_1 = \{F_t\}_{t \geq 0}$ *and $S_2 = \{G_t\}_{t \geq 0}$ are both groups of parabolic automorphisms of Δ, then condition (i) of the theorem holds automatically, so the commutativity of F_1 and G_1 implies that S_1 and S_2 commute.*

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